



## FOUNDATIONS OF MATHEMATICS

### TRUTH AND MEANING, COMPLETENESS AND CONSISTENCY

Seminar Leader: **Rodrigo Ferreira da Rosa**

House Leader: **Nanako Ito**

## Course Description

What does it mean for a statement to be true? What about provable? Are there statements that are true but not provable? If truth can only be captured by intelligence, does mathematics not capture truth, or is mathematics actually the first Artificial Intelligence we created? In this seminar, we will attack these questions by proving two incompleteness theorems discovered by the Austrian mathematician Kurt Gödel in 1931. They roughly assert that any attempt to formalize all of mathematics in an axiomatic system will fail in being simultaneously complete and consistent. We will study what implications these results have on our understanding of Truth, and how they rely on the elusive nature of Meaning.

## Schedule

### Day 1: *What Is Truth in Mathematics?*

In the first day, we look at some mathematical fallacies that raise the question of what is a mathematical proof. This will naturally lead us to the need of a formal system to separate such proofs from non-proofs. We look at Euclidean and non-Euclidean geometry to see that a formal system is not the absolute incarnation of Truth, but rather is allowed variations to conform to different Meanings. We introduce zeroth-order logic (a.k.a. “propositional calculus”) and first-order logic, which is the basis of Number Theory. With a formalization of Number Theory under our belt, we will be able to state the Incompleteness theorems. Homework: The MU-challenge, read *Canon by intervallic Augmentation*, p.153, Salviati and Simplicio’s dialogue on page 408.

### Day 2: *What is Meaning in Mathematics?*

In the second day, we fully describe a formal system for doing Number Theory called Typographic Number Theory, or TNT for short. We discuss how meaning emerges as a result of an *isomorphism*, and whether isomorphisms exist by themselves or are the workings of intelligence. Homework: express “ $a$  is a power of  $k$ ” in TNT, read *Contracrostipunctus*, p. 75.

### Day 3: *Famous Paradoxes and Gödel’s Theorem*

In the third day, students will get a chance to explain some big paradoxes and math results to the class themselves. Examples are: the barber paradox, autological and heterological words, Epimenides paradox, Hilbert hotel, and Cantor’s diagonalization argument. Using these ideas, we will construct Gödel’s G-string in TNT which proves the incompleteness theorems. Homework: read *Little Harmonic Labyrinth*, p. 103, and *The Magnificrab, Indeed*, p. 549 and chapter XVII, p. 559.



**Day 4: Consistency and Completeness**

In the final day, we will delve further in the implications of the incompleteness theorems. We will look at consistency and completeness, at  $\omega$ -consistency and at recursive structures and processes. We will look at recent advances in computer science and Artificial intelligence, at Church-Thesis and see that “tangled recursion is at the heart of intelligence.” We will ask ourselves to what extent Number Theory is also a form of AI.

## Pre-Seminar Assignments and Readings

- Readings: Excerpts from Douglas Hofstadter’s *Gödel, Escher, Bach* (GED); will be provided in Japanese translation as well
- Day 1 Homework: *Gödel, Escher, Bach* (GED) p. 153, p. 408, the MU-challenge
- Day 2 Homework: *Gödel, Escher, Bach* (GED) p. 75, express ‘ $a$  is a power of  $k$ ’
- Day 3 Homework: *Gödel, Escher, Bach* (GED) p. 103, p. 549, p. 559; prepare paradox presentation

## Message from the House Leader

Hello, my name is Nanako and I’m a rising junior at the College of Wooster, a small liberal arts college in Ohio. I am majoring in Computer Science and Math, which is a big change considering that I had been an arts and humanities student in Japan. In my free time I love watching movies (mostly drama films) and I’m a fan of Haruki Murakami novels. I also enjoy having one-on-one conversations with different people, learning about who they are and what they are interested in. I hope that KSS will expose you to different ways of thinking and help you get to know people you wouldn’t meet otherwise. I look forward to getting to know all of you! If you have any questions before KSS, please don’t hesitate to email me.

## Message from the Seminar Leader

Hi! My name is Rodrigo and I am a junior at Yale University, where I am working towards a dual bachelor’s and master’s degree in Mathematics. Aside from sciences in general, I am really interested in languages and I have lived in six different countries. In my free time, I like to bike, hike, practice martial arts, sing and of course talk about math with the younger generation. I am excited to meet you all!